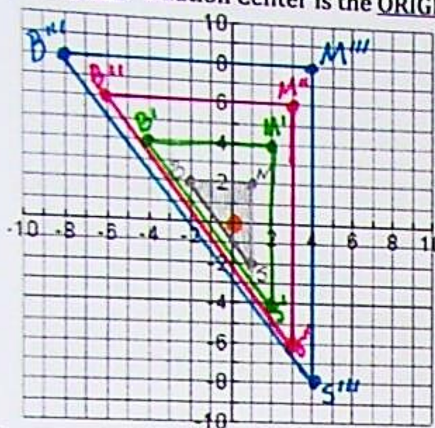
enVision Topic 7

7-1 NOTES Dilations, Center of Dilation, & Scale Factor

A dilation is a transformation that results in a similar figure.

Angle measures are preserved (≅) and side lengths are proportional.

Example 1: Dilation Center is the ORIGIN. GRAPH: Graph Pre-Image ΔBMS with coordinates B(-2, 2), M(1, 2), S(1, -2).



A) Dilate with Center of Dilation (0, 0) and scale factor 2.

sides & perimeter double Area = 2^2, or quadruple

B) Dilate with Center of Dilation (0, 0) and scale factor 3.

sides & perimeter triple Area = 3^2, nine times +

C) Dilate with Center of Dilation (0, 0) and scale factor 4.

sides & perimeter quadruple Area = 4^2 16 times bigger

D) List the side lengths of each triangle, then find the perimeter and area.

	Pre-Image	Scale Factor = 2	Scale Factor = 3	Scale Factor = 4
Sides	3 – 4 – 5	6 – 8 – 10	9 – 12 – 15	12 - 16 - 20
Perimeter	12u	24u	36u	48u
Area	$\frac{1}{2}(12)$ $6u^2$	$\frac{1}{2}(48)$ $24u^2$	$\frac{1}{2}(108)$ $54u^2$	$\frac{1}{2}(192)$ $96u^2$

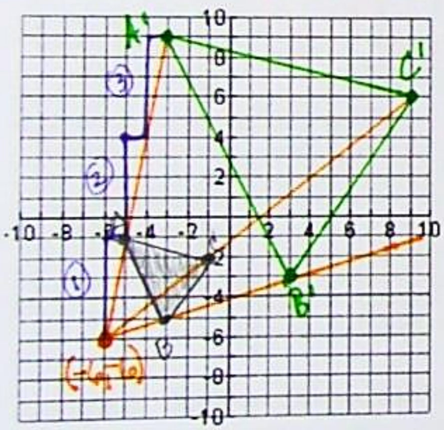
E) Write a ratio comparing ΔA'B'C' to ΔABC for each resulting perimeter and area.

Perimeters: $\frac{24}{12} = 2$ $\frac{36}{12} = 3$ $\frac{48}{12} = 4$ Areas: $\frac{24}{6} = 4$ $\frac{54}{6} = 9$ $\frac{96}{6} = 16$

F) How does each resulting side length, perimeter, and area compare to the corresponding measures of the Pre-Image.

Sides and Perimeters increase by scale factor. Area increases by $(\text{scale factor})^2$

Example 2: Dilation Center a point, NOT the origin! ... and thinking about SLOPE!

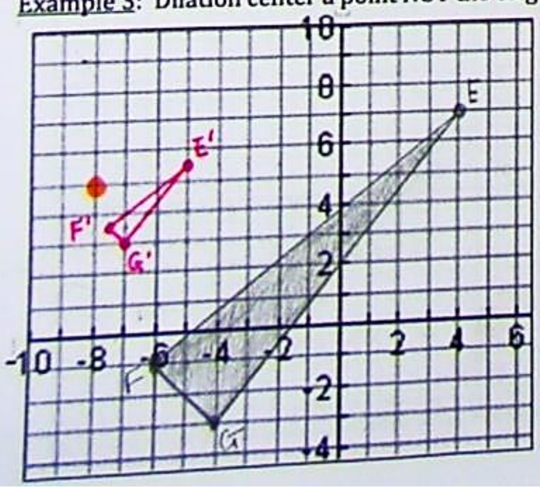


GRAPH: ΔABC with coordinates A(-5, -1), B(-3, -5), C(-1, -2).
Dilate with Center of Dilation (-6, -6) and scale factor 3.

Question) How is slope related to the vertices of the pre-image and resulting image under a dilation?

The distance of the image point from its corresponding pre-image point is three times the rise and three times the run from the center of dilation to each vertex of the resulting image.

Example 3: Dilation center a point NOT the origin, scale factor a fraction.



GRAPH: ΔEFG with coordinates E(4, 7), F(-6, -1), G(-4, -3).
Dilate with Center of Dilation (-8, 5) and scale factor $\frac{1}{4}$.

F	G	E
↑$\frac{6}{4}$ ← $\frac{2}{4}$	↑$\frac{8}{4}$ ← $\frac{4}{4}$	↓$\frac{2}{4}$ ← $\frac{12}{4}$
↓1.5 → $\frac{1}{2}$	↓2 → 1	↑$\frac{1}{2}$ → 3

Question 1) How does a fractional scale factor affect the size of a figure?

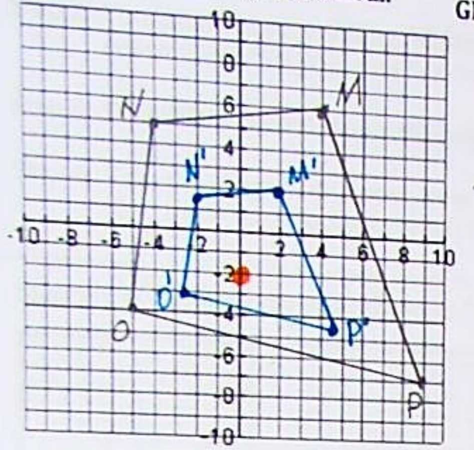
The figure shrinks – aka: reduction

Question 2) How would you describe the direction that the figure projected

Under a reduction, the figure moves towards the center of dilation

Another way to think of a dilation is either a <u>stretch</u> or a <u>shrink</u>, which in technical terms are called <u>enlargement</u> or <u>reduction</u>.

Example 4: Dilate a quadrilateral.

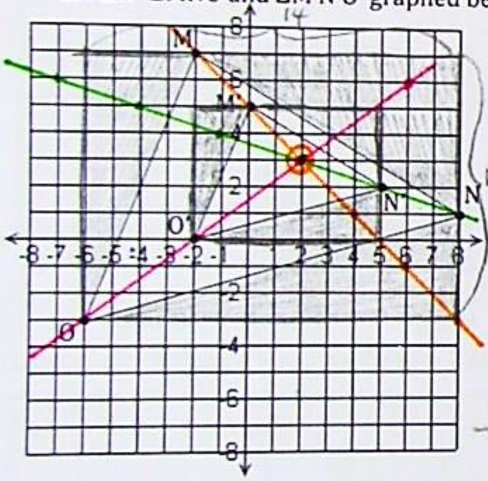


GRAPH: MNOP with coordinates M(4, 6), N(-4, 5), O(-5, -4), and P (9, -7). Dilate with Center of Dilation (0, -2) and scale factor $\frac{1}{2}$.

M	N	O	P
$\uparrow\frac{8}{2}\rightarrow\frac{4}{2}$	$\uparrow\frac{7}{2}\leftarrow\frac{4}{2}$	$\downarrow\frac{2}{2}\leftarrow\frac{5}{2}$	$\downarrow\frac{5}{2}\rightarrow\frac{9}{2}$
$\frac{4\uparrow}{2\rightarrow}$	$\frac{3.5\uparrow}{2\leftarrow}$	$\frac{1\downarrow}{2.5\leftarrow}$	$\frac{2.5\downarrow}{4.5\rightarrow}$

Example 5: Determine the center of dilation and scale factor, given a pre-image and its image, & finding AREA.

GIVEN: ΔMNO and ΔM'N'O' graphed below.



<u>Center of Dilation</u>

(2,3)

<u>Scale Factor</u>

center to N center to N'

$\frac{-2}{6}\frac{\div 2}{\div 2}\ \frac{-1}{3}$ $\boxed{SF: \frac{1}{2}}$

Question 1) What should be the ratio of the area of the Image to the Pre-Image?

$\frac{1}{4} : 1$ or $1 : 4$ $\left(\frac{1}{4}\right)$

Question 2) How could you find the area of each triangle using the "enclosure method"?

see next page

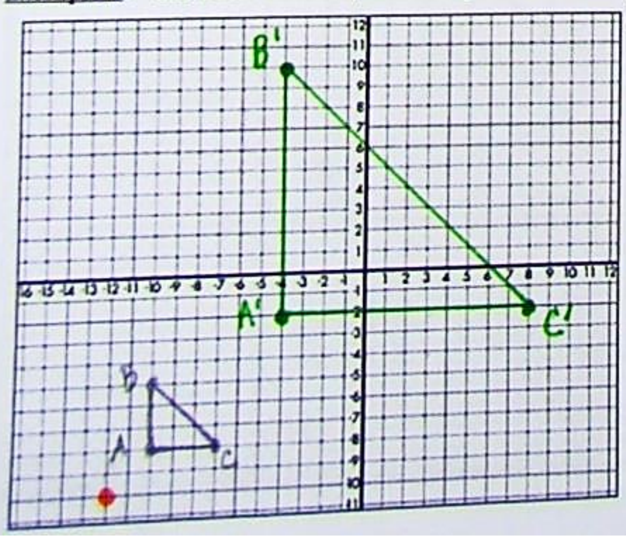
$(10)(14) = 140u^2 - 78u^2 = \boxed{62u^2}$ | $(7)(5) = 35 - 19.5 = \boxed{15.5u^2}$

$\frac{1}{2}(4)(14) = 28$ $\frac{1}{2}(4)(10) = 20$ $\frac{1}{2}(6)(10) = 30$ | $5 + 7 + 7.5$

Question 3) Show whether your answer to Question 1 is correct.

$\frac{new}{old}$ $\frac{15.5}{62} = \frac{31}{124} = \frac{1}{4}$ or 0.25

Example 6: One more dilation, then compare areas!



GRAPH: ΔABC with coordinates A(-10, -8), B(-10, -5), C(-7, -8). Dilate with Center of Dilation (-12, -10) and scale factor 4.

$\overline{BC} = \sqrt{3^2+3^2} = \sqrt{2 \cdot 9} = 3\sqrt{2}$ $\overline{B'C'} = \sqrt{12^2+12^2} = \sqrt{2 \cdot 144} = 12\sqrt{2}$

A) Find the perimeter of each triangle, then write a ratio of perimeter comparing that of ΔA'B'C to ΔABC.

$P_{\triangle ABC} = 3+3+3\sqrt{2} = 6+3\sqrt{2}$

$P_{\triangle A'B'C'} = 12+12+12\sqrt{2} = 24+12\sqrt{2}$

$\frac{24+12\sqrt{2}}{6+3\sqrt{2}} = \frac{12(2+\sqrt{2})}{3(2+\sqrt{2})}$ (4)

B) Find the area of each triangle, then write a ratio of the area of ΔA'B'C to ΔABC.

$A_{\triangle ABC} = \frac{1}{2}(3 \cdot 3) = \frac{1}{2}(9) = 4.5$

$A_{\triangle A'B'C'} = \frac{1}{2}(12 \cdot 12) = \frac{1}{2}(144) = 72$

$\frac{72}{4.5} = 16$

C) What relationship exists between the ratios and the scale factor used in this problem?

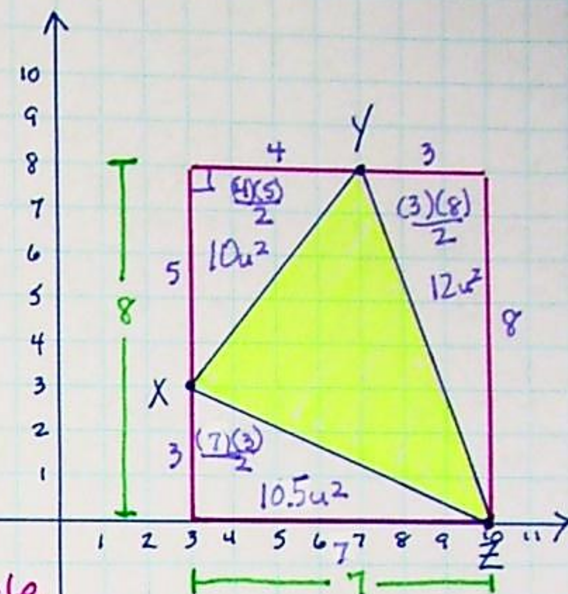
ratio of perimeters = scale factor

ratio of areas = (scale factor)^2

Enclosure Method : Find the area of ΔXYZ

1) If a triangle (or other figure) doesn't "square" with the grid lines, then "enclose" the triangle with a rectangle.

   * Be sure each vertex of the triangle is "on" a side of the rectangle.



2) Calculate the area of the rectangle.

$$A = bh = (7)(8) = \boxed{56u^2}$$

3) Find the area of each RIGHT triangle (which represents the "extra" area), and add them: $10 + 12 + 10.5 = \boxed{32.5u^2}$

4) The area of the triangle (ΔXYZ) is the difference between the rectangle area minus the RIGHT triangles' area sum. ($\Delta + \Delta + \Delta$)

So, subtract: $56 - 32.5 = \boxed{23.5u^2}$

(rectangle) − (Rt Δ's) = Area of ΔXYZ

Derive Mapping RULES for dilations, Center not Origin: Applying the Math to Pre-Image Coordinates

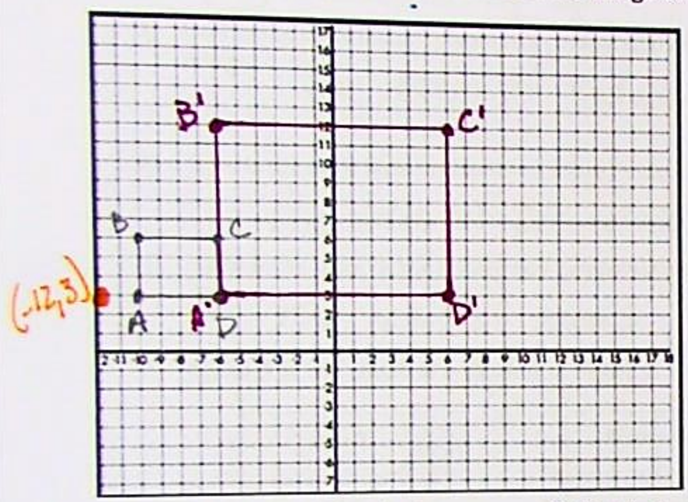
Step 1: Dilating from the origin is easier! Move the point that is the center of dilation back to the origin by adding the opposite of its x and y values to the point's coordinates.

Step 2: You must keep everything relative! Move the vertices of the Pre-image by applying the same values to the coordinates that you used to relocate the center of dilation.

Step 3: DILATE the figure! Apply the scale factor to the resulting coordinates of each vertex.

Step 4: Take the image back home! Now restore the center of dilation and the figure to their original position. How? Undo the relocation values by adding the opposite of the values applied to them in steps 1 & 2.

Example 7: Applying the steps above to Pre-image coordinates.



GRAPH: ABCD with coordinates A(-10, 3), B(-10, 6), C(-6, 6), D(-6, 3).

A) Dilate with Center of Dilation (-12, 3) and scale factor 3.

$A(\overset{+12}{-10}, \overset{-3}{3}) \to (3\cdot 2, 3\cdot 0) \to (\overset{-12}{6}, \overset{+3}{0}) = A'(-6,3)$

$B(\overset{+12}{-10}, \overset{-3}{6}) \to (3\cdot 2, 3\cdot 3) \to (\overset{-12}{6}, \overset{+3}{9}) = B'(-6,12)$

$C(\overset{+12}{-6}, \overset{-3}{6}) \to (3\cdot 6, 3\cdot 3) \to (\overset{-12}{18}, \overset{+3}{9}) = C'(6,12)$

$D(\overset{+12}{-6}, \overset{-3}{3}) \to (3\cdot 6, 3\cdot 0) \to (\overset{-12}{18}, \overset{+3}{0}) = D'(6,3)$

B) What is the ratio of the sides?

$\frac{A'B'}{AB} = \frac{9}{3} : \boxed{\frac{3}{1}}$ $\frac{B'C'}{BC} = \frac{12}{4} = \boxed{\frac{3}{1}}$

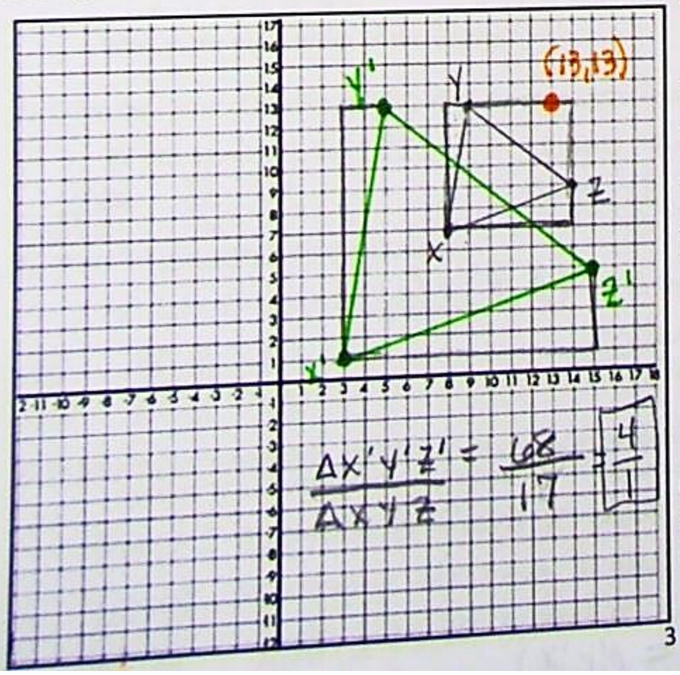
C) What is the ratio of the perimeters?

$\frac{A'B'C'D'}{ABCD} = \frac{2(9+12)}{2(3+4)} = \frac{21}{7} : \boxed{\frac{3}{1}}$

D) What is the ratio of the areas?

$\frac{A'B'C'D'}{ABCD} = \frac{{}^{3}\cancel{12}\cdot\cancel{9}^{3}}{{}_{1}\cancel{4}\cdot\cancel{3}_{1}} = \frac{3\cdot 3}{1} = \frac{3^2}{1} = \boxed{\frac{9}{1}}$

Example 8: Writing a Mapping Rule for Dilations.



GRAPH: ΔXYZ with coordinates X(8, 7), Y(9, 13), Z(14, 9).

Dilate with Center of Dilation (13, 13) and scale factor 2.

$X(\overset{-13}{8}, \overset{-13}{7}) \to (2\cdot -5, 2\cdot -6) \to (\overset{+13}{-10}, \overset{+13}{-12}) = X'(3,1)$

$Y(\overset{-13}{9}, \overset{-13}{13}) \to (2\cdot -4, 2\cdot 0) \to (\overset{+13}{-8}, \overset{+13}{0}) = Y'(5,13)$

$Z(\overset{-13}{14}, \overset{-13}{9}) \to (2\cdot 1, 2\cdot -4) \to (\overset{+13}{2}, \overset{+13}{-8}) = Z'(15,5)$

A) How could you use the slopes of a pair of corresponding sides to verify the scale factor ... or to determine it if it was unknown?

slope $\overline{XZ}$ $m = \frac{2}{6}$ rise $2 \to 4$ *2

slope $\overline{X'Z'}$ $m = \frac{4}{12}$ run $6 \to 12$ *2 $\boxed{SF=2}$

B) Use the enclosure method to find the area of each triangle, and then find the ratio of areas comparing ΔX'Y'Z' to ΔXYZ.

$h = 13-1 = 12$ $144-(12+24+40)$ | $h=6$ $36-(6+3+10)$

$b = 15-3 = 12$ $144-76 = 68$ | $b=6$ $36-19 = 17$

C) Given coordinates (x,y), scale factor 5 and center of dilation (-2, 4), write an algebraic rule for this dilation.

$(x'y') = [5(x+2)-2, 5(y-4)+4]$

Follow Up Questions!

1) How can you determine whether a figure has undergone an enlargement or a reduction given the coordinates of a point and its image?

Example 1: X (15, 35) and X' (3, 7)

Smaller values result, ⇒ reduction

Example 2: A(2, 3) and A'(8, 12)

larger values result, ⇒ enlargement

2) If you know the coordinates of a pair of corresponding vertices of a figure's Pre-image and Image, how can they be used to find the scale factor?

Example 1: Use coordinates above.

Scale factor: $\frac{new}{old}$ $\quad \frac{3}{15}$ $\frac{7}{35}$ $\quad \frac{1}{5}$ $\frac{1}{5}$ ⟵ $\frac{1}{5}$

Example 2: Use coordinates above.

scale factor: $\frac{new}{old}$ $\quad \frac{8}{2}$ $\frac{12}{3}$ $\quad 4 \quad 4$ ⟵ 4

3) How is the scale factor related to a comparison of segment lengths when writing a ratio of <u>SIDES</u> of the Image to Pre-Image?

The ratio of sides $\frac{new}{old}$ = scale factor

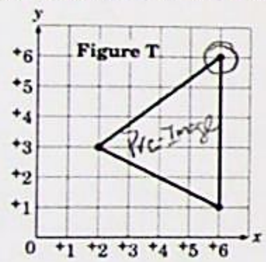
4) How is the scale factor related to a comparison of the <u>PERIMETERS</u> in a ratio of Image to Pre-Image?

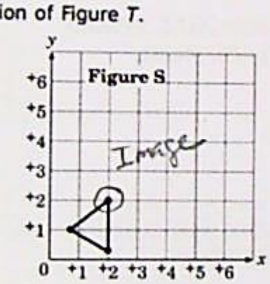
The ratio of perimeters $\frac{new}{old}$ = scale factor

5) How is the scale factor related to a comparison of the <u>AREAS</u> in a ratio of Image to Pre-Image?

The ratio of areas $\frac{new}{old}$ = (scale factor)^2

6) Figure *S* is the result of a dilation of Figure *T*.





$\frac{new}{old}$ $\quad \frac{(2,\ 2)}{(6,\ 6)}$ reduce!

scale factor = $\frac{1}{3}$

What is the scale factor of the dilation?

7) Given the center of dilation is (0, 0), state the <u>scale factor</u> applied to the following pairs of corresponding vertices:

A) P(3, 4) → P'(12, 16) <u>4</u> $\quad \frac{new}{old}$ $\frac{12}{3}$ $\frac{16}{4}$

B) L(-15, 50) → L'(-3, 10) <u>$\frac{1}{5}$</u> $\quad \frac{new}{old}$ $\frac{-3}{-15}$ $\frac{10}{50}$

C) D(1, 9) → D'(6, 54) <u>6</u> $\quad \frac{new}{old}$ $\frac{6}{1}$ $\frac{54}{9}$

8) Write the <u>general rule</u> for the dilation.

A) G(21, 6) → G'(7, 2) $(x, y) \rightarrow (\frac{x}{3}, \frac{y}{3})$

Scale factor $\frac{7}{21}$ $\frac{2}{6}$ = $\frac{1}{3}$

B) N(2, 15) → N'(4, 30) $(x, y) \rightarrow (2x, 2y)$

Scale factor $\frac{4}{2}$ $\frac{30}{15}$ = 2

9) **Recap:** In your own words, explain how to calculate the coordinates for the vertices of an image under a dilation with scale factor "k" and with center of dilation a point that is not the origin. (hint: You may use point P(a, b) as the center of dilation)

First: subtract a from x and b from y.
Second: multiply the new x and y by k.
Third: add a to the x-value and add b to the y-value.
Fourth: Use prime symbols for the Image coordinate point.

$(x', y') = k(x-a),\ k(y-b)$
$= (kx-ka)+a,\ (ky-kb)+b$

Algebraically